

T-duality for open strings

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The T-duality transformations between open and closed superstrings in different D-manifolds are generalized to curved backgrounds with commuting isometries. We address some global aspects like the occurrence of orientifold boundaries in general sigma models, higher genus world sheets, and the case of non-compact isometries. The various world-volume effective actions are shown to transform properly under T-duality. We also include a brief discussion of the canonical transformations of boundary states in the operator formalism.

1. Introduction

Target space T-duality is one of the most intriguing symmetries in string theory. It was initially discovered as an invariance of toroidal compactifications of closed strings under the change of the space time radius, R to $\frac{1}{R}$ [1], but it was soon realized that it is a much more generic property, and indeed a systematic technique allows the study of this symmetry in all backgrounds with non-trivial group of isometries [2][3][4] [5].

While most earlier work dealt with closed strings only, thought at the time the most interesting ones from the physical point of view, recent work on string-string duality (cf., for example, [6] [7]), shows that there is a hidden unity between different types of string theories. In fact, already at a perturbative level, open strings can in a sense be considered as closed strings in world-sheet orbifolds [8]. The action of T-duality on open string theories has been studied in the past [9] [10]. Recent excitement followed Polchinski's idea that some exotic objects appearing in the dual of Type I theory (*D-branes*) are the carriers of R-R charges needed for the implementation of the S-duality conjecture [11]. An excellent recent review can be found in [12].

The aim of the present paper is to examine some aspects of these problems in more detail. The use of semiclassical methods allows us to derive many of the well established facts about open string T-duality, in a simple and unified treatment and, most importantly, provide an appropriate framework for the generalization to arbitrary spacetime backgrounds with (commuting) isometries, even when an exact conformal field theory description is not available. We also pay due attention to the effects of non-orientable world-sheets in Type I theories, as well as the subtleties of the T-duality/supersymmetry interplay [13].

The paper is organized as follows. In section 2 we review the auxiliary gauging procedure and apply it to the open string/D-brane duality in the bosonic case, with a detailed discussion of the mapping of boundary conditions, and the emergence of collective coordinates for the D-brane in this formalism. In section 3 we discuss some new features arising in the supersymmetric case, concerning the fermion boundary conditions. In section 4 we check that the low energy effective world volume actions (Dirac-Born-Infeld and Chern-Simons) have the desired covariance properties under T-duality. Finally, we consider the

higher genus case in appendix A. Appendix B is a review of facts about T-duality for non-compact isometries, and appendix C contains a very brief account of the canonical transformations involved in T-duality with boundaries.

2. Open bosonic strings in background fields

Let us begin with the simplest bosonic model, which, while allowing for the most interesting physical phenomena, is devoid of complications due to supersymmetry. We shall consider open and closed bosonic strings propagating in an arbitrary d -dimensional metric and (abelian) gauge field. Wess-Zumino antisymmetric tensors are not consistent if the theory is non-orientable, but we shall include them nevertheless for the time being. In modern language, we have a closed string interacting with a Dirichlet $(d - 1)$ -brane, and we consider non-trivial massless backgrounds in the longitudinal directions.

In the neutral case (that is, the charge is opposite in both ends of the string), the action can be written as¹:

$$S = \frac{1}{4\pi} \int_{\Sigma} (g_{\mu\nu} \eta^{ab} + i b_{\mu\nu} \epsilon^{ab}) \partial_a x^{\mu} \partial_b x^{\nu} + \frac{i}{2\pi} \int_{\partial\Sigma} n_a A_{\mu} \partial_b x^{\mu} \epsilon^{ab}. \quad (2.1)$$

The classification of allowed world-sheet topologies is much more complicated in the open case than in the more familiar closed one. The Euler characteristic can be written, in a somewhat symbolic form, as

$$\chi = 2 - 2g - c - b, \quad (2.2)$$

where g is the number of handles, b the number of boundaries, and c the number of crosscaps. To the lowest order of string perturbation theory $\chi = 1$, only contribute the disc D_2 and in the non-orientable case the crosscap or, to be more precise, the two-dimensional real projective plane $P_2(R)$. (To the following “one loop” order, corresponding to $\chi = 0$, we have the annulus A_2 , the Möbius band M_2 , and the Klein bottle K_2). In this section we shall only consider the leading contributions from the disc and the crosscap.

¹ We set $\alpha' = 1$ throughout.

The action (2.1) will be invariant under a target isometry with Killing vector k^μ ,

$$\delta_\epsilon x^\mu = \epsilon k^\mu(x) \quad (2.3)$$

provided a vector ω_μ and a scalar φ exist, such that:

$$\begin{aligned} \mathcal{L}_k g_{\mu\nu} &= 0 \\ \mathcal{L}_k b_{\mu\nu} &= \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \\ \mathcal{L}_k A_\mu &= -\omega_\mu + \partial_\mu \varphi, \end{aligned} \quad (2.4)$$

where \mathcal{L}_k represents the Lie derivative with respect to the Killing vector.

In the neutral case, it is clear that the boundary term representing the coupling of the background gauge field to the open string can be incorporated in the bulk action through the simple substitution ²

$$b_{\mu\nu} \rightarrow B_{\mu\nu} = b_{\mu\nu} + F_{\mu\nu}. \quad (2.5)$$

Using the conditions on the background fields, it is easy to show that

$$\mathcal{L}_k B_{\mu\nu} = 0. \quad (2.6)$$

In order to perform the duality transformation, it is convenient to rewrite (2.1) as a redundant gauge system where the isometry (2.3) is gauged. We must introduce a Lagrange multiplier to ensure that the auxiliary gauge field is flat. According to (2.6), minimal coupling is enough to construct the gauged action:

$$\begin{aligned} S_{\text{gauged}} &= \frac{1}{4\pi} \int_\Sigma (g_{\mu\nu} \eta^{ab} + i B_{\mu\nu} \epsilon^{ab}) D_a x^\mu D_b x^\nu \\ &\quad + \frac{i}{4\pi} \int_\Sigma \tilde{x}^0 (\partial_a V_b - \partial_b V_a) \epsilon^{ab} - \frac{i}{2\pi} \int_{\partial\Sigma} \tilde{x}^0 V, \end{aligned} \quad (2.7)$$

where $D_a x^\mu = \partial_a x^\mu + k^\mu V_a$ and, in adapted coordinates the Killing vector reads $k = \frac{\partial}{\partial x^0}$. The one-form associated to the gauge field is represented as $V = V_a dx^a$. The rôle of the boundary term is to convey invariance under translations of the Lagrange multiplier: $\tilde{x}^0 \rightarrow \tilde{x}^0 + C$.

² When the charges at both ends of the string do not add to zero, one cannot get rid completely of the boundary term. We shall comment on a similar situation later in the main text.

Boundary conditions are restricted by several physical requirements. The gauge parameter must have the same boundary conditions as the world sheet fields (i.e. Neumann), in order for the isometry to be realized on the boundary of the Riemann surface. This has the obvious consequence that, if the gauge $V = 0$ were needed, it would be necessary to impose on the gauge fields the boundary condition $n^a V_a \equiv V_n = 0$ (because this component can never be eliminated with gauge transformations obeying Neumann boundary conditions). It turns out, however, that in order to show the equivalence of (2.7) with the original model (2.1), the behavior of $V|_{\partial\Sigma}$ is immaterial. The only way the action (2.7) can now lead to the *unique* restriction $dV = 0$ on the gauge field is to restrict the variations of the Lagrange multiplier in such a way that $\delta\tilde{x}^0|_{\partial\Sigma} = 0$. In this way we are forced to impose Dirichlet boundary conditions $\tilde{x}^0 = C$ on the multiplier. Since the rest of the coordinates remain Neumann, a Dirichlet $(d-2)$ -brane is obtained. Besides, this ensures gauge invariance of (2.7).

The last two terms in (2.7) can be combined into

$$\frac{i}{2\pi} \int_{\Sigma} -d\tilde{x}^0 \wedge V. \quad (2.8)$$

This means that the gauge field enters only algebraically in the action, and it can be replaced by its classical value (performing the gaussian integration only modifies the dilaton terms):

$$V_a^{c\ell} = -\frac{1}{k^2} (k^\mu g_{\mu\nu} \partial_a x^\nu + i\epsilon_a{}^b \partial_b \tilde{x}^0 + i\epsilon_a{}^b k^\mu B_{\mu\nu} \partial_b x^\nu). \quad (2.9)$$

Particularizing now to adapted coordinates, $k = \frac{\partial}{\partial x^0}$ and choosing the dual gauge $x^0 = 0$, we get the dual model, whose functional form is exactly like in (2.1), but with the backgrounds $\tilde{G}_{\mu\nu}$, $\tilde{B}_{\mu\nu}$ given in terms of the original ones through Buscher's formulas:

$$\begin{aligned} \tilde{G}_{00} &= \tilde{g}_{00} = \frac{1}{g_{00}} \\ \tilde{G}_{0i} &= \frac{B_{0i}}{g_{00}} \\ \tilde{G}_{ij} &= g_{ij} - \frac{g_{0i}g_{0j} - B_{0i}B_{0j}}{g_{00}} \\ \tilde{B}_{0i} &= \tilde{b}_{0i} = \frac{g_{0i}}{g_{00}} \\ \tilde{B}_{ij} &= B_{ij} - \frac{g_{0i}B_{0j} - B_{0i}g_{0j}}{g_{00}}. \end{aligned} \quad (2.10)$$

In deriving these expressions, some care must be exercised in choosing the appropriate variables in adapted coordinates. In this frame, the isometry is represented by simple translations $x^0 \rightarrow x^0 + \epsilon$. This means that the various backgrounds must be independent of x^0 , up to target space gauge transformations, which in this model are defined by

$$b \rightarrow b + d\lambda \quad A \rightarrow A - \lambda, \quad (2.11)$$

where λ is an arbitrary one-form, in such a way that $B = b + dA$ is invariant. This gauge ambiguity is responsible for the occurrence of the non-trivial Lie derivatives in (2.4). In order to consistently reach the gauge $x^0 = 0$ in the closed string sector (world sheets without boundaries), the torsion Lie derivative must be cancelled within the local patch of adapted coordinates. In fact, it is easily seen that the gauge transformation λ , defined as

$$\mathcal{L}_k \lambda = -\omega + d\varphi$$

cancels both the ω and φ terms in (2.4). In this gauge, all fields are locally independent of x^0 .

The behavior of the dilaton under T-duality is always a subtle issue. In the present situation this is even more so, due to the fact that the metric is not a massless background of the open string, and one has to consider closed string corrections, thus driving the sigma model away from the conformally invariant point, in order to get a consistent Fischler-Susskind mechanism (cf. [14] and references therein). There is, however, a necessary condition for the equivalence of the two theories, and this is that the effective action must remain invariant. It turns out that this condition is sufficient to determine the dual dilaton to the value

$$\tilde{\phi} = \phi - \frac{1}{2} \log k^2. \quad (2.12)$$

The invariance under translations of the dual model can sometimes be put to work in our benefit. Let us consider, for simplicity, the Wilson line $A_0 = \text{diag}(\theta_1, \dots, \theta_N)$, when only the coordinate x^0 is compactified in a circle of length $2\pi R$ in an otherwise flat background. The Wilson line itself, in the sector of winding number n is given by

$$\sum_{a=1}^N e^{2\pi n R \theta_a i}$$

This term is reproduced in the dual model by simply taking into account the “total derivative term” coming from the gaussian integration of the auxiliary gauge field, namely

$$\int_{\Sigma} dx^0 \wedge d\tilde{x}^0,$$

which by Stokes’ theorem can be written as $-\oint \tilde{x}^0 dx^0 = -C 2\pi n R$, where C is the constant value of the multiplier on the boundary. This value can depend on the (implicit) Chan-Paton indices of the world sheet fields, and we recover with our techniques the result of [9][11][12] that the Wilson line considered induces in the dual model a series of D-branes with fixed positions determined by the θ_a parameters. A more detailed discussion, generalized to arbitrary genus world sheets can be found in appendix A.

An interesting observation is that the collective motion of the D-brane is already encoded in Buscher’s formulas. To see this, notice that the dual backgrounds in (2.10) differ from the standard duals without gauge fields by the terms:

$$\begin{aligned}\tilde{G}_{0i} &= \tilde{g}_{0i} - \tilde{g}_{00} \partial_i A_0 \\ \tilde{G}_{ij} &= \tilde{g}_{ij} + \tilde{g}_{00} \partial_i A_0 \partial_j A_0 - \tilde{g}_{0i} \partial_j A_0 - \tilde{g}_{0j} \partial_i A_0 \\ \tilde{B}_{ij} &= \tilde{b}_{ij} + F_{ij} + \tilde{b}_{0i} \partial_j A_0 - \tilde{b}_{0j} \partial_i A_0.\end{aligned}\tag{2.13}$$

Using these formulas it is easily checked that the dual sigma model reduces to the standard one in terms of the backgrounds $\tilde{g}_{\mu\nu}$ and $\tilde{b}_{\mu\nu}$, provided we make the replacement

$$\tilde{x}^0 \rightarrow \tilde{x}^0 + A_0(x^i).\tag{2.14}$$

Thus, the gauge field component A_0 acquires the dual interpretation of the transverse position of the D-brane, as a function of x^i , which become longitudinal world volume coordinates³. The same result follows from careful consideration of the boundary conditions.

³ It is amusing to notice that, if we choose to cancel only the ω dependence in (2.4), then the previous formulas still hold if we substitute A_0 by $A_0 - \varphi$. In addition, we have $\partial_0 A_j = \partial_j \varphi$. If we were so bold as to interpret x^0 as a timelike coordinate, we could regard the A_j configuration as a dyonic field.

2.1. Boundary Conditions

It is well known [14] that in the presence of nontrivial backgrounds the Neumann boundary condition, which physically is equivalent to the absence of momentum flow through the edges of the string, is modified (because the physical momentum is modified accordingly), to

$$g_{\mu\nu}\partial_n x^\nu + iB_{\mu\nu}\partial_t x^\nu = 0, \quad (2.15)$$

where the vector n and t are the normalized outer normal and tangent vector to the world sheet boundary and, correspondingly, $\partial_n \equiv n^a \partial_a$, and $\partial_t \equiv t^b \partial_b$.

It is not immediately clear what are the correct generalizations of (2.15) for the gauged action (2.7). We claim that the good choice is:

$$g_{\mu\nu}D_n x^\nu + iB_{\mu\nu}D_t x^\nu = 0. \quad (2.16)$$

First of all, if we use the fact that the gauge connection is flat, $V_a = \partial_a \alpha$, and perform an isometric transformation $x^\mu \rightarrow x^\mu + \alpha k^\mu$, then (2.16) easily yields (2.15). On the other hand, (2.16) reduces, in adapted coordinates and in the $x^0 = 0$ gauge, to

$$\begin{aligned} g_{00}V_n + g_{0j}\partial_n x^j + iB_{0j}\partial_t x^j &= 0 \\ g_{i0}V_n + g_{ij}\partial_n x^j + iB_{i0}V_t + iB_{ij}\partial_t x^j &= 0. \end{aligned} \quad (2.17)$$

If now the classical gauge fields $V_a^{c\ell}$ are plugged in (2.17), and use is made of Buscher's formulas, the following dual boundary conditions are obtained:

$$\begin{aligned} \partial_t \tilde{x}^0 &= 0 \\ \tilde{G}_{i\mu}\partial_n \tilde{x}^\mu + i\tilde{B}_{i\mu}\partial_t \tilde{x}^\mu &= 0. \end{aligned} \quad (2.18)$$

where we denote $\tilde{x}^\mu = (\tilde{x}^0, x^j)$. It is also possible to verify involution ($T^2 = 1$). The dual model is automatically written in adapted coordinates to the dual Killing $\tilde{k} = \frac{\partial}{\partial \tilde{x}^0}$. A further T-duality transformation can then be performed with respect to it. Substituting again the values of the dual classical gauge fields, $\tilde{V}_a^{c\ell}$, in the dual covariant boundary conditions, the correct ones for the original model (2.15) are recovered.

The mixed boundary conditions (2.18) correspond to a Dirichlet $(d-2)$ -brane in the dual background. To check this, notice that for a D-brane of general shape given by the equations

$$\tilde{x}^\mu = y^\mu(x^j),$$

we can define induced metric and antisymmetric tensor backgrounds in the usual fashion

$$(\tilde{g}_{ij})_{\text{induced}} = \partial_i y^\mu \tilde{g}_{\mu\nu} \partial_j y^\nu, \quad (2.19)$$

and similarly for $\tilde{b}_{\text{induced}}$. As expected, the previously defined \tilde{G}_{ij} and \tilde{B}_{ij} in (2.13) are precisely the induced world volume backgrounds under the identification $y^0 = -A_0$, $y^j = x^j$. Furthermore, shifting the dual coordinate $\tilde{x}^0 \rightarrow \tilde{x}^0 + A_0$ we obtain the dual boundary conditions in the form of the second reference in [9]:

$$\begin{aligned} \tilde{x}^0 &= y^0(x^j) \\ \partial_i y^\mu \tilde{g}_{\mu\nu} \partial_n \tilde{x}^\nu + i \tilde{B}_{i\nu} \partial_t \tilde{x}^\nu &= 0. \end{aligned} \quad (2.20)$$

In the case of unoriented strings, the change from Neumann to Dirichlet conditions is supplemented by an orbifold projection in the space-time which reverses the orientation of the world sheet (the orientifold). This result follows easily from the T-duality mapping at the level of the conformal field theory (see [9]): $\partial_z x(z) \rightarrow \partial_z x(z)$, $\partial_{\bar{z}} x(\bar{z}) \rightarrow -\partial_{\bar{z}} x(\bar{z})$. In order to study this question in a curved background, let us consider the lowest order unoriented topology, namely the crosscap $P_2(R)$. It is enough for our purposes to make in the boundary of the unit disc the identification of opposite points: $x^\mu(\sigma) = x^\mu(\sigma + \pi)$. (Here $\sigma \in (0, 2\pi)$ just parametrizes the boundary.) Dropping the zero mode, this yields the conditions [14]:

$$\begin{aligned} \partial_n x^\mu(\sigma) &= -\partial_n x^\mu(\sigma + \pi) \\ \partial_t x^\mu(\sigma) &= \partial_t x^\mu(\sigma + \pi). \end{aligned} \quad (2.21)$$

When gauging the isometry, there is a covariant generalization of these conditions, namely

$$\begin{aligned} D_n x^\mu(\sigma) &= -D_n x^\mu(\sigma + \pi) \\ D_t x^\mu(\sigma) &= D_t x^\mu(\sigma + \pi). \end{aligned} \quad (2.22)$$

Using the value of $V_a^{c\ell}$ obtained above, we easily find, after fixing the $x^0 = 0$ gauge:

$$D_n x^0 = -\frac{1}{k^2} \partial_n x^i k_i + i \partial_t \tilde{x}^0. \quad (2.23)$$

The antisymmetric tensor and abelian gauge field backgrounds are projected out from the physical spectrum of unoriented strings in the weak field limit, and so we only consider a nontrivial metric background. Then (2.22) and (2.23) yield the dual boundary conditions in the form:

$$\left(i \partial_t \tilde{x}^0 - \frac{1}{k^2} k_j \partial_n x^j \right) (\sigma + \pi) = - \left(i \partial_t \tilde{x}^0 - \frac{1}{k^2} k_j \partial_n x^j \right) (\sigma).$$

The terms containing the Killing vector cancel away owing to the boundary conditions of the original model; the rest reduces to the orientifold condition on \tilde{x}^0 :

$$\begin{aligned} \partial_t \tilde{x}^0(\sigma) &= -\partial_t \tilde{x}^0(\sigma + \pi) \\ \partial_n \tilde{x}^0(\sigma) &= \partial_n \tilde{x}^0(\sigma + \pi). \end{aligned} \quad (2.24)$$

The first equation implies $\tilde{x}^0(\sigma) + \tilde{x}^0(\sigma + \pi) = \text{constant}$, so that the crosscap is embedded as a twisted state of the orbifold. This is one of the main results of this paper; it implies that the orbifold character of the dual target space is a generic phenomenon, and not a curious peculiarity of toroidal backgrounds. It is curious to remark that the dual manifold always enjoys parity $\tilde{x}^0 \rightarrow -\tilde{x}^0$ as an isometry, because $\tilde{g}_{0i} = 0$.

The rest of the coordinates still satisfy standard crosscap conditions. An important consequence of (2.24) is that at least two points of the boundary are mapped to the orientifold fixed points in the target, which means that local contributions of non orientable world sheets are concentrated at the orientifold location; in the bulk of space-time the dual theory is orientable along the direction \tilde{x}^0 [9]. This is compatible with the appearance of a non vanishing dual antisymmetric tensor $\tilde{b}_{0i} = g_{0i}/g_{00}$ as long as the original background has a “boost” component. The effects of this background field are suppressed only for world sheets mapped to the fixed point. Another observation is that, in the absence of a $U(1)$ gauge field, there is no collective coordinate for the orientifold, which becomes a rigid object. Indeed, according to the formulas (2.13) and (2.19), the induced backgrounds are exactly the same as the vacuum dual backgrounds.

3. Superstrings

The extension of the previous results to the supersymmetric case is for the most part straightforward. In particular, the dual background formulas (2.10) still apply. There are, however, some subtleties associated with the choice of fermion boundary conditions in the presence of non trivial torsion or abelian gauge fields. We find that, generically, T-duality induces complicated boundary conditions mixing fermions and bosons.

We shall examine these problems for the general $(1, 1)$ supersymmetric sigma model in the lowest order world sheets (the disk and the projective plane). The action is

$$S = -\frac{1}{2\pi} \int d^2z d^2\theta (g_{\mu\nu} + B_{\mu\nu}) D_+ X^\mu D_- X^\nu. \quad (3.1)$$

where the superfields are $X^\mu(\theta_+, \theta_-, z, \bar{z}) = x^\mu + \theta_+ \psi_+^\mu + \theta_- \psi_-^\mu + \theta_+ \theta_- F^\mu$, the superderivatives being $D_+ = \partial_+ + \theta_+ \partial_z$ and $D_- = \partial_- + \theta_- \partial_{\bar{z}}$. When expanded in components, (3.1) contains the standard interaction terms of the general $(1, 1)$ sigma model with torsion, and also a fermionic boundary action of the form

$$S_{\text{line}} = \frac{i}{4\pi} \oint B_{\mu\nu} (\psi_+^\mu \psi_+^\nu + \psi_-^\mu \psi_-^\nu). \quad (3.2)$$

The gauge component (recall that $B = b + dA$) is the well known fermionic part of the supersymmetric (abelian) Wilson line.

The model (3.1), with d -dimensional target space would appear in the discussion of type II strings with Dirichlet $(d - 1)$ -branes in general (longitudinal) background fields. Also, if we drop the torsion and abelian gauge fields $B = 0$ we may consider a type I string in curved space⁴. In order to keep the discussion as simple as possible, we start with generalized boundary conditions of Neumann type in all coordinates in (3.1), and define the mixed boundary conditions at the dual $(d - 2)$ -brane as those induced by T-duality.

⁴ In type I superstrings, the only consistent backgrounds are Wilson lines of $SO(32)$, which need a different treatment (see appendix A for a discussion of some simple cases). In particular, the dual contains an orientifold and 16 D-branes. It is important, however to consider general backgrounds with torsion because even when it is zero in the initial theory, it is generically induced in the dual one after a T-duality transformation.

In the bosonic sector, Neumann boundary conditions are defined enforcing the complete boundary equations of motion. However, in the presence of fermionic boundary actions such as (3.2), this requirement might be too strong, since it would produce trivial quantum dynamics of the boundary fermions.

This is easily illustrated in the simple situation where $g_{\mu\nu} = \delta_{\mu\nu}$ and $B_{\mu\nu} = b_{\mu\nu}$ is constant. Using matrix notation, the full action splits into a *surface* and a *line* term

$$\mathcal{L}_{\text{surface}} = -\partial_z x^t (1+b) \partial_{\bar{z}} x + \psi_+^t \partial_{\bar{z}} \psi_+ + \psi_-^t \partial_z \psi_- \quad (3.3)$$

$$\mathcal{L}_{\text{line}} = \frac{1}{2} \partial_{\bar{z}} (\psi_+^t b \psi_+) - \frac{1}{2} \partial_z (\psi_-^t b \psi_-). \quad (3.4)$$

It is important to notice that $\mathcal{L}_{\text{surface}}$ enjoys invariance under chiral rotations of the form $\psi_- \rightarrow \mathcal{O} \psi_-$ provided $\mathcal{O}^t \mathcal{O} = 1$ is orthogonal. If we want to promote this to a symmetry of the full action (including the *line* term), then we have to demand in addition that $[\mathcal{O}, b] = 0$.

The variation of the full action induces the following boundary terms

$$\oint \delta x^t ((1-b) \partial_z x - (1+b) \partial_{\bar{z}} x) + \delta \psi_-^t (1+b) \psi_- - \delta \psi_+^t (1-b) \psi_+, \quad (3.5)$$

which lead to the standard [14] conditions on the bosonic sector,

$$\partial_z x = \frac{1+b}{1-b} \partial_{\bar{z}} x. \quad (3.6)$$

However, neither the usual NS-R conditions $\psi_+ = \eta \psi_-$, $\eta = \pm 1$, nor the “rotated” ones

$$\psi_+ = \eta \frac{1+b}{1-b} \psi_-, \quad (3.7)$$

solve (3.5). Notice that (3.7) is equivalent to the normal NS-R conditions by means of the above mentioned chiral rotation symmetry, with $\mathcal{O} = (1+b)/(1-b)$. The boundary conditions (3.6) and (3.7) are the ones considered in [14], using the operator formalism.

An explicit solution of (3.5) is given by the ansatz

$$\psi_+ = \eta \frac{1+b}{1-b} \mathcal{J} \psi_- \quad (3.8)$$

provided $\mathcal{J}^t (1 - b) \mathcal{J} = (1 + b)$. The matrix \mathcal{J} can be chosen satisfying $\mathcal{J}^t = \mathcal{J}$, $\mathcal{J}^2 = 1$, and $\{\mathcal{J}, b\} = 0$. If b is block-diagonalized in the form $b = \text{diag}_j(ib_j \sigma_2)$, then $\mathcal{J} = \text{diag}_j(\sigma_3)$ (notice the resemblance with an (almost) complex structure in the target). Given that $(1 - b)/(1 + b)$ is orthogonal and commutes with b , the boundary conditions (3.8) are a chiral rotation of $\psi_+ = \eta \mathcal{J} \psi_-$, which explicitly break the $O(d)$ Lorentz symmetry, even in the limit of vanishing electromagnetic background.

The important point is that with the “fully classical” boundary conditions (3.8)

$$S_{\text{line}} \sim \oint (\psi_+^t b \psi_+ + \psi_-^t b \psi_-) = 0,$$

as corresponds to an on-shell first order system. Therefore, in order to have some quantum fermionic dynamics on the boundary, we may set to zero only those total derivatives coming from the variation of $\mathcal{L}_{\text{surface}}$, letting $\mathcal{L}_{\text{line}}$ freely fluctuate. In this way we are led to the standard NS-R conditions in the fermionic sector, which we can rotate by a convenient orthogonal matrix, thus recovering (3.7). The particular combination of the (1,1) world sheet supersymmetry which is preserved depends on such rotations. The variation of the complete action under transformations generated by the currents $J_+ = -\psi_+^t \partial_z x(z)$, and $J_- = -\psi_-^t \partial_{\bar{z}} x(\bar{z})$ has the form

$$\delta_{\text{ss}} S \sim \oint (\epsilon_+ \psi_+^t - \epsilon_- \psi_-^t) (1 - b) \partial_z x,$$

where we have used the bosonic boundary conditions (3.6). We see that the NS-R conditions $\psi_+ = \eta \psi_-$ ensure invariance under the supersymmetry generated by $J_+ + \eta J_-$, whereas the rotated ones (3.7) preserve instead the conveniently rotated current

$$J_{\text{rot}} = -\psi_+^t \partial_z x - \eta \psi_-^t \frac{1 - b}{1 + b} \partial_{\bar{z}} x.$$

These currents define the induced supersymmetry on the boundary. Note that the surviving boundary supersymmetry is not quite the same as the combination that maps the fermionic and bosonic conditions into each other. For example, $J_+ + \eta J_-$ maps (3.7) into (3.6).

For curved backgrounds, a very general class of boundary conditions with consistent T-duals is given by:

$$\begin{aligned} (g - B)_{\mu\nu} \partial_z x^\nu - (g + B)_{\mu\nu} \partial_{\bar{z}} x^\nu + W_\mu(\psi_-) &= 0 \\ \psi_+^\mu &= R^\mu(\psi_-). \end{aligned} \tag{3.9}$$

The functionals W and R are regarded as power series expansions in the right moving fermions ψ_-^μ , with coefficient functions satisfying the isometry constraint (independent of x^0). There are two particular solutions of special interest, corresponding to the two classes of boundary conditions considered previously. With the choice $W_\mu = 0$ and $R^\mu(\psi_-) = \eta \psi_-^\mu$, we recover the standard Neumann NS-R conditions which we would derive neglecting the boundary variations of (3.2). On the other hand, a set of “fully classical” boundary conditions (including (3.2)), can be obtained setting $R^\mu(\psi_-) = \eta \mathcal{J}_\nu^\mu \psi_-^\nu$, and $W_\mu(\psi_-) = \Omega_{\mu\nu\rho} \psi_-^\nu \psi_-^\rho$, with the constraints

$$\mathcal{J}_\alpha^\mu (g - B)_{\mu\nu} \mathcal{J}_\beta^\nu = (g + B)_{\alpha\beta},$$

$$\Omega_{\mu\nu\rho} = B_{\mu[\nu;\rho]} + B_{\alpha\mu;\beta} \mathcal{J}_{[\nu}^\beta \mathcal{J}_{\rho]}^\alpha + B_{\alpha\beta} \mathcal{J}_{[\nu}^\alpha \mathcal{J}_{\rho];\mu}^\beta.$$

It is easily checked that these twisted conditions still satisfy $S_{\text{line}} = 0$.

Now the standard algorithm for T-duality goes through in the superfield formalism. The action is invariant under the super-isometry transformation $\delta_\epsilon X^\mu = \epsilon k^\mu$, provided the background fields satisfy the same conditions as in the bosonic case. The auxiliary gauging is achieved again by minimal coupling to the gauge superfields V_+ and V_- , whose transformation laws under the gauged isometry are $\delta_\epsilon V_\pm = -D_\pm \epsilon$, where now ϵ is a $(1,1)$ bosonic superfield. The other components of the supergauge multiplet are written in terms of V_\pm as $V_z = D_+ V_+$ and $V_{\bar{z}} = D_- V_-$. The supercovariant derivatives are $\nabla_\pm X^\mu = D_\pm X^\mu + V_\pm k^\mu$ and $\nabla_{z(\bar{z})} X^\mu = \partial_{z(\bar{z})} X^\mu + V_{z(\bar{z})} k^\mu$. The gauged action with Lagrange supermultiplier term is

$$S_{\text{gauged}} = -\frac{1}{2\pi} \int d^2 z d^2 \theta \left((g_{\mu\nu} + B_{\mu\nu}) \nabla_+ X^\mu \nabla_- X^\nu - D_+ \tilde{X}^0 V_- - D_- \tilde{X}^0 V_+ \right). \quad (3.10)$$

Integrating out the supermultiplier enforces the super-flatness condition $D_+ V_- + D_- V_+ = 0$, ensuring the equivalence of (3.10) and (3.1). If the gauge superfields are integrated instead, the dual model follows after gauge fixing $X^0 = 0$.

The gauged boundary conditions, equivalent to (3.9), can be written as

$$\begin{aligned} g_{\mu\nu} (\nabla_z - \nabla_{\bar{z}}) X^\nu - B_{\mu\nu} (\nabla_z + \nabla_{\bar{z}}) X^\nu + W_\mu (\nabla_- X) |_{\theta=0} &= 0 \\ \nabla_+ X^\mu &= R^\mu (\nabla_- X) |_{\theta=0}. \end{aligned} \quad (3.11)$$

The transformation to the dual variables proceeds as usual by fixing the gauge $X^0 = 0$ in adapted coordinates. The final result is simply obtained by the substitutions $\partial_z x^0 \rightarrow D_+ V_+^{c\ell}$, $\partial_{\bar{z}} x^0 \rightarrow D_- V_-^{c\ell}$, and $\psi_{\pm}^0 \rightarrow V_{\pm}^{c\ell}$ in eq. (3.9), with the following classical gauge supermultiplet:

$$D_{\pm} V_{\pm}^{c\ell}|_{\theta=0} = \pm \partial_i \tilde{k}_j^{\mp} \psi_{\pm}^i \psi_{\pm}^j \pm \partial_i \tilde{k}^2 \psi_{\pm}^i \tilde{\psi}_{\pm}^0 \pm \tilde{k}_i^{\mp} \partial_{z(\bar{z})} x^i \pm \tilde{k}^2 \partial_{z(\bar{z})} \tilde{x}^0$$

$$V_{\pm}^{c\ell}|_{\theta=0} = \pm \tilde{k}_i^{\mp} \psi_{\pm}^i \pm \tilde{k}^2 \tilde{\psi}_{\pm}^0,$$

where $\tilde{k}_{\mu}^{\pm} \equiv \tilde{k}^{\nu} (g \pm B)_{\nu\mu}$. One finds a complicated set of mixed Dirichlet-Neumann conditions, with the same purely bosonic terms as in (2.18), and extra boson-fermion interactions. A rather non trivial check, which is met by these duality mappings, is provided by the involution property: $T^2 = 1$.

Explicit (somewhat complicated) expressions for the dual functionals \tilde{R}^{μ} and \tilde{W}_{μ} can be read off from the previous manipulations in all generality. Here we will just quote the explicit form of the simplest (and perhaps more interesting) boundary conditions: the Neumann NS-R case (that is $W = 0$ and $R = \eta$ in (3.9)),

$$\begin{aligned} \tilde{\psi}_+^i &= \eta \tilde{\psi}_-^i \\ \tilde{\psi}_+^0 + \eta \tilde{\psi}_-^0 &= -2\eta k_j^* \tilde{\psi}^j \\ (\partial_z + \partial_{\bar{z}}) \tilde{x}^0 &= -2\partial_i k_j^* \tilde{\psi}_-^i \tilde{\psi}_-^j \\ \tilde{g}_{i\mu}(\partial_z - \partial_{\bar{z}}) \tilde{x}^{\mu} - \tilde{B}_{i\mu}(\partial_z + \partial_{\bar{z}}) \tilde{x}^{\mu} &= 2k_i^* \partial_j \tilde{k}^2 \tilde{\psi}_-^j \tilde{\psi}_-^0 - 2(\tilde{k}_i^+ \partial_j k_l^* - k_i^* \partial_j \tilde{k}_l^+) \tilde{\psi}_-^j \tilde{\psi}_-^l \end{aligned} \quad (3.12)$$

where we denote $k_i^* \equiv \tilde{k}_i/\tilde{k}^2 = B_{0i}$, and $\tilde{\psi}^i = \psi^i$. Notice that the non trivial fermionic terms are all proportional to the “electric field” B_{0i} , and therefore correspond to a certain D-brane shape and state of motion in the dual language. In particular, they are absent for the case of type I/8-brane (with orientifold) duality. Using (2.6) and the Bianchi identities in the pure electromagnetic case $B = dA$, it is easy to check that the fermionic terms drop from the Dirichlet condition of the multiplier \tilde{x}^0 , in the third equation of (3.12). The relative minus sign in the induced spin structure of the fermions $\tilde{\psi}_{\pm}^0$ accounts for the reversal of space-time chirality: if the closed string sector of the original model is type A, then the T-dual is type B.

In the case of the type I string we have to consider the projective plane to leading order. In fact, the absence of torsion in the original background makes the discussion much simpler. Let us denote by $P = \sigma$ and $P' = \sigma + \pi$ the two identified points on the boundary of the disk. The standard crosscap bosonic conditions (2.21), together with $\psi_+^\mu(P') = \eta \psi_-^\mu(P)$, are equivalent to the covariant ones

$$\begin{aligned}\nabla_+ X^\mu(P') &= \eta \nabla_- X^\mu(P)|_{\theta=0} \\ (\nabla_z - \nabla_{\bar{z}})X^\mu(P') &= (\nabla_z - \nabla_{\bar{z}})X^\mu(P)|_{\theta=0} \\ (\nabla_z + \nabla_{\bar{z}})X^\mu(P') &= -(\nabla_z + \nabla_{\bar{z}})X^\mu(P)|_{\theta=0}.\end{aligned}\tag{3.13}$$

In the gauge $X^0 = 0$ they reduce to

$$\begin{aligned}\tilde{\psi}_+^0(P') &= -\eta \tilde{\psi}_-^0(P) \\ (\partial_z - \partial_{\bar{z}})\tilde{x}^0(P') &= -(\partial_z - \partial_{\bar{z}})\tilde{x}^0(P) \\ (\partial_z + \partial_{\bar{z}})\tilde{x}^0(P') &= (\partial_z + \partial_{\bar{z}})\tilde{x}^0(P),\end{aligned}\tag{3.14}$$

getting in this way the orientifold conditions for the boson as well as the corresponding change of sector for the fermion, the remaining equations being unaltered.

It would be interesting to study the interplay between (3.12) and space-time supersymmetry in the context of (2, 2) Kähler sigma models. In particular, for isometries acting without fixed points, we expect no subtleties in the T-duality mapping of target space supersymmetry charges [13]. The total or partial supersymmetry breaking induced by the “boost” terms in (3.12) (proportional to k_j^*), reflects again the presence of non vanishing “electric fields” B_{0i} in the original model.

4. T-duality of the effective world volume actions

In the closed string sector, the T-duality transformations (2.10), together with (2.12), leave the low energy effective action invariant (in the string frame). Accordingly, in the presence of open string backgrounds, T-duality should produce directly the Dirac-Born-Infeld action appropriate for the corresponding D-brane. We may illustrate this fact for the

simplest case. We start from the ten dimensional open string effective action, to leading order in a derivative expansion:

$$S_{\text{eff}} = \int \frac{d^{10}x}{\alpha'^5} e^{-\phi} \sqrt{\det(g + b + F)_{\mu\nu}}. \quad (4.1)$$

For the unoriented type-I string, $b = 0$ and F lies in $SO(32)$, so that a trace over group indices is understood.

We will consider factorized Chan Paton factors such that A_0 is diagonal and A_i is independent of x^0 and assumes a block factorized form according to the eigenspaces of A_0 . Then $F_{0j} = \partial_0 A_j - \partial_j A_0 = -\partial_j A_0 = \partial_j y$. The basic identity we need is the following, which holds for an arbitrary matrix $M_{\mu\nu}$:

$$\det(M_{\mu\nu}) = M_{00} \det \left(M_{ij} - \frac{M_{i0} M_{0j}}{M_{00}} \right).$$

Applying the formula to the determinant above we find

$$\det(g + b + F)_{\mu\nu} = g_{00} \det[(\tilde{g} + \tilde{b})_{ij} + F_{ij} + \tilde{g}_{00} \partial_i y \partial_j y + \partial_i y (\tilde{g} + \tilde{b})_{0j} + \partial_j y (\tilde{g} - \tilde{b})_{0i}]$$

where $\tilde{g}_{\mu\nu}$ and $\tilde{b}_{\mu\nu}$ are the standard vacuum dual fields. Using (2.13) and (2.19), we easily obtain the world volume action for the dual 8-brane:

$$\int \frac{d^{10}x}{\alpha'^5} e^{-\phi} \sqrt{\det(g + b + F)_{\mu\nu}} = 2\pi \int \frac{d^9x}{\alpha'^{9/2}} e^{-\tilde{\phi}} \sqrt{\det((\tilde{g} + \tilde{b})_{\text{induced}} + F)_{ij}}, \quad (4.2)$$

where we have used the relations $\int dx^0 = 2\pi\sqrt{\alpha'}$ and $\tilde{\phi} = \phi - \log \sqrt{g_{00}}$. Notice that the dilaton transformation is essential to cancel the Kaluza-Klein factor of the compact volume, so that we obtain the correct dimensionality for the world-volume action. We see that the results of [9] follow immediately from those of [14] using T-duality.

In the context of type I and II strings, we have also R-R p -forms A_p , whose transformation law under duality is not obvious from the sigma model construction, because R-R backgrounds involve vertex operators with non-trivial spin field and super-ghost dependence. It is possible to guess their transformation laws requiring T-duality of effective low energy effective action [15]. Here we just check the T-duality covariance of the generalized

D-brane Chern-Simons actions in the simplified case where $g_{0i} = B_{0i} = 0$. The universal coupling of the R-R forms to the p -brane is [14] [16] [17]

$$S_{\text{source}}^{RR} = (2\pi\sqrt{\alpha'})^{3-p} \int_{\Sigma_{p+1}} \sum_{q=1}^{p+1} A_q \text{Str} e^{b+F+dy}, \quad (4.3)$$

up to corrections in derivatives of the NS-NS field strength $B = b + F$. “Str” is an antisymmetrized trace over gamma matrices: we expand the exponent with the structure $\exp \gamma^\mu \gamma^\nu B_{\mu\nu}$ and antisymmetrize all the traces over gammas. The leading term is the usual $\int_{\text{world-volume}} A_{p+1}$, in units $A_{p+1} \sim \text{length}^{-4}$. The sum over q in (4.3) runs over odd numbers for type IIA theory and over even numbers for type IIB and type I (in fact, for type I there are only one, five and nine branes⁵). Since T-duality interchanges A and B types, the T-duality transformation must be, to leading order:

$$A_{\mu_1, \dots, \mu_p, 0} = \frac{1}{\sqrt{g_{00}}} \tilde{A}_{\mu_1, \dots, \mu_p}.$$

So, a $p+1$ even form in IIB or type I goes into a p odd form in type IIA or type I' . With this form of the duality transformation we ensure T-invariance of the kinetic term,

$$\frac{1}{(p+2)!} \int d^{10}x \sqrt{g} |F_{p+2}|^2 = \frac{1}{(p+1)!} \int d^{10}x \sqrt{\tilde{g}} |\tilde{F}_{p+1}|^2,$$

and the correct mapping of the source terms.

$$\int_{\Sigma_{p+1}} A_{p+1} = 2\pi\sqrt{\alpha'} \int_{\Sigma_p} \tilde{A}_p.$$

It is important to point out that the duality formulas in this section, particularly (4.2), do not take into account the back reaction on the background fields caused by the presence of the D-brane. In that case, an inhomogeneous profile in the x^0 direction is generated for the various fields, including the dilaton. Such non-trivial dependence on x^0 spoils the isometry property, and the standard T-duality formulas do not apply. These back reaction effects are important in order to resolve some paradoxes in the type I/Heterotic

⁵ Several consistency requirements, most notably modular invariance and anomaly cancellation, restrict the general structure of D-manifolds. See, for example [18].

string duality [7]. Large sources for the dilaton appear as large uncancelled tadpoles in string perturbation theory. Such tadpoles are homogeneous throughout space-time in the type I theory, and inhomogeneous in the T-dual language, where there is an orientifold in place. It would be interesting to understand the T-duality transformation of the corrected backgrounds, even if they do not enjoy an isometry symmetry. In other words, we would like to understand T-duality in the presence of a non-trivial Fischler-Susskind mechanism. A simple observation to make is that the isometry property was used in (4.2) only to represent the right hand side as a nine-dimensional integral. If the x^0 integral is not simply factorized in the compact volume, eq. (4.2) still holds in the sense of an “average” over the compact dimension.

5. Conclusions

Generic T-duality transformations of curved D-manifolds with commuting isometries have been studied, and Buscher’s formulas have been shown to follow, including the collective motions of the D-branes in the dual target space. We have also considered in detail the T-duality between Neumann and Dirichlet boundary conditions in general sigma models of bosonic or supersymmetric (1,1) type. Orientifolds, in particular, have been obtained as generic phenomena, not tied up to specific aspects of toroidal compactifications.

We have deferred to the appendices a number of issues of formal nature. These include the consideration of higher genus world sheets, albeit with the simplest backgrounds only (we can see no obstruction in principle, however to repeat the analysis in the general case). Some aspects of very simple Wilson lines have also been studied, although the T-duality transformation of the most general non-abelian Wilson line remains as an interesting open problem.

An appendix on T-duality in general, in the case in which the orbits of the isometry group are non-compact, has been included for pedagogical purposes, and some specific examples have been worked out in detail, although the results here are not new (cf. [19]). Finally, we have outlined the canonical formalism for T-duality with boundaries in the flat background.

Much work remains to be done in the analysis of supersymmetric sigma models, specially those with extended supersymmetry, and the interplay with space-time supersymmetry. It is also important to study the effects of generic non-abelian gauge backgrounds, as well as space-time dependent dilaton sources, whose consideration seems to be necessary to really understand the appearance of the strong coupling regime preventing a disproof of Heterotic/Type I string/string duality in [7].

Acknowledgements We have benefited from discussions with L. Alvarez-Gaumé, C. Bachas, C. Callan, C. Gómez, Y. Lozano and P. Townsend. The work of E. A. is supported by the grant AEN/93/0673. J.L.F.B. is supported by NSF PHY90-21984 grant. J.B. enjoys a CAM fellowship.

Appendix A. Global issues

In this appendix we give a path integral derivation of the well known duality between D-brane location and Chan-Paton factors, with careful consideration of the technicalities regarding the translational zero modes and global holonomies, both in the world sheet and in the target space. In order to simplify notation we will consider a flat decoupled background $g_{0i} = b_{0i} = 0$ and concentrate on the higher genus path integral for the field x^0 which we denote by X throughout this section.

A.1. Higher genus

It will be instructive to follow the opposite route to the one taken in section 2, and start with a Dirichlet path integral for a genus g Riemann surface with N boundaries B_1, \dots, B_N mapped respectively to points y_1, \dots, y_N in the target circle of radius R (some of the y_i could be identical). The relevant amplitude is given by the periodic sum

$$\Gamma^g(R; y_1, \dots, y_N) = \sum_{m_j} \Gamma_{\text{fixed}}^g(R; y_j - y_1 + 2\pi R m_j), \quad (\text{A.1})$$

where

$$\Gamma_{\text{fixed}}^g(R; y_1, \dots, y_N) = \sum_n \int_{X_{B_j=y_j}} \mathcal{D}X_n e^{-\frac{1}{4\pi} \int (dX_n)^2}. \quad (\text{A.2})$$

Because of the translational invariance of the action, the path integral only depends on the differences of y_j , say $y_j - y_1$, ($j = 2, \dots, N$), and we may take as well $y_1 = 0$. This fact was taken into account in (A.1) by considering only $N - 1$ independent periodic sums. In (A.2), the X_n field is multivalued around handles as $\oint_a dX_n = 2\pi R n_a \in 2\pi R \mathbf{Z}$, where $a = 1, \dots, 2g$ labels a homology basis of the Riemann surface.

The path integral proof proceeds as usual by transforming the measure to a first order formalism via the change of variables $\partial_\alpha X \rightarrow V_\alpha$ (equivalently, gauging the isometry $X \rightarrow X + \epsilon$ and fixing the gauge $X = 0$). The transformation of the measure is

$$\begin{aligned} \mathcal{D}X &= \prod dX \prod_{B_j} \delta(X - y_j) \\ &= \prod dV \prod ' \delta(*dV) \prod_{a=1}^{2g} \delta\left(\oint_a V\right) \prod_{j=2}^N \delta\left(\int_{\gamma_j} V - y_j + y_1\right) \prod_{B_j} \delta(V). \end{aligned} \quad (\text{A.3})$$

Here γ_j is a system of paths from a reference point where we fix the translational zero mode, to the boundary components. It is convenient to choose the reference point to lie at one particular boundary, say B_1 . Then $X_{CM} = y_1$, and we have $N - 1$ paths from a distinguished point on B_1 to the rest of the boundary components. Once we have chosen this point, it is useful to drop the corresponding integration variable X_{CM} at the expense of the delta function $\delta(X_{CM} - y_1)$. We denote this suppression of one integration variable by a prime in ultralocal products. In a lattice regularization:

$$\mathcal{D}'X = \prod_i 'dX_i \prod_{B_j} ' \delta(X_i - y_j),$$

where i denote the points in the lattice. Following [20], the complete measure (A.3) can be motivated in the lattice as follows. The first three factors in the second line of (A.3) account for the standard change of variables from X_i to $V_{ij} = X_i - X_j$. The flatness constraint $*dV = 0$ is represented in the lattice as $\sum_{\partial(\text{faces})} V = 0$. Since the change of variables is linear we only have to check that we have the same number of degrees of freedom. Prior to Dirichlet fixing we have $N_0 - 1$ integrals over points in the lattice, N_1 link integrals and $N_2 - 1 + N + 2g$ delta functions, and the matching is ensured by Euler's theorem: $N_0 - 1 = N_1 - N_2 + 1 - N - 2g$. Now we have to express the Dirichlet deltas

in first order form (in terms of the link variables). Pick a path γ in the lattice from the reference point $X_{CM} = y_1$ to some boundary $B \neq B_1$, and label the point variables on that boundary by $X_0, X_1, \dots, X_k, \dots$. Then, the relations $X_k = y_1 + V_{k,k-1} + \dots + V_{10} + \sum_{\gamma} V$ can be used iteratively to prove the delta function identity:

$$\delta \left(\sum_B V \right) \prod_{i \in B} \delta(X_i - y_B) = \delta \left(\sum_{\gamma} V - y_B + y_1 \right) \prod_B \delta(V). \quad (\text{A.4})$$

For the remaining B_1 boundary we have

$$\prod_{B_1} ' \delta(X_i - y_1) = \delta(V_{10}) \delta(V_{21}) \cdots \delta(V_{mm-1})$$

so that, together with $\delta(\sum_{B_1} V)$, we get a total factor of $\prod_{B_1} \delta(V)$, and we have thus succeeded in reducing all terms in the measure to first order form.

Now we can solve the constraints on the dual lattice defining $V_{ij} = V_{IJ}$ if $(ij)^* = (IJ)$, where I denote points in the dual lattice, that is faces of the original lattice. Then we introduce the Lagrange multiplier field:

$$\prod_{\text{faces}, I} \delta \left(\sum_{\partial I} V \right) = \prod_I \int \frac{d\tilde{X}_I}{2\pi} e^{i\tilde{X}_I \sum_{J(I)} V_{IJ}},$$

and similarly for the rest of the terms in the measure. In continuum notation we exponentiate the holonomy constraints as

$$\delta \left(\oint_a V \right) = \int_{\mathbf{R}} \frac{d\ell_a}{2\pi} e^{i\ell_a \oint_a V} = \int_{\mathbf{R}} \frac{d\ell_a}{2\pi} e^{i\ell_a \int V \wedge h_a},$$

where h_a is a basis for homology one-forms $\oint_a h_b = \delta_b^a$. We can define similar one forms for each of the contours γ_j , such that $\int_{\gamma_j} V = \int V \wedge h_j$. They can be written as $h_j = d\alpha_j$, where α_j is an angular variable centered at $X_{CM} = 0$, taking values 1 and 0 “above” and “below” γ_j . In particular, their integrals along the boundaries are $\oint_{B_j} h_j = \oint_{B_j} d\alpha_j = \pm 1$, depending on the orientation of the contour integral. So the h_j are a basis of “winding modes” of the open string boundaries, and can be used to exponentiate the global part of the Dirichlet constraints:

$$\delta \left(\int_{\gamma_j} V - y_j + y_1 \right) = \prod_{j=2}^N \int_{\mathbf{R}} \frac{dp_j}{2\pi} e^{-ip_j(y_j - y_1)} e^{ip_j \int V \wedge h_j}.$$

Now we can do the V integral. First, we shift $V \rightarrow V - 2\pi n_a h_a$ and obtain the factor

$$\sum_{n_a} e^{-2\pi i n_a \ell_a R} = R^{-2g} \sum_{n_a} \delta\left(\ell_a - \frac{n_a}{R}\right), \quad (\text{A.5})$$

where we have used $\int h_a \wedge h_b = \delta_{ab}$. This results in a quantization of ℓ_a in units of the dual radius $1/R$. The remaining gaussian path integral to evaluate is

$$\prod_{j=2}^N \int \frac{dp_j}{2\pi} e^{-ip_j(y_j - y_1)} \sum_n \int_{t_{B_j} \cdot V=0} \mathcal{D}' \tilde{X}_n \mathcal{D}V e^{-\frac{1}{4\pi} \int V^2 + \frac{i}{2\pi} \int V \wedge (d\tilde{X}_n + 2\pi p_j h_j)},$$

where $\oint_a d\tilde{X}_n = \frac{2\pi}{R} n_a$ is multivalued in the dual circle of radius $1/R$. The saddle point of this integral is at $V_{cl} = -i * (d\tilde{X} + 2\pi p_j h_j)$. Accordingly, the Dirichlet boundary conditions $t_B \cdot V_{cl} = 0$ are mapped into Neumann conditions for the fluctuating \tilde{X} field $t_B \cdot *dX_{cl} = n_B \cdot d\tilde{X}_{cl} = 0$.

The periodic sum in (A.1) has the effect of discretizing the p_j via

$$\sum_{m_j} e^{-2\pi i R m_j p_j} = R^{-N+1} \sum_{m_j} \delta\left(p_j - \frac{m_j}{R}\right). \quad (\text{A.6})$$

In this way we obtain the final result for T-duality of Dirichlet path integrals at any genus:

$$\Gamma^g(R; y_j) = \text{const.} \times R^{-2g-N+1} \sum_{m_j} \delta_{\sum m} e^{-iy_j m_j / R} \tilde{\Gamma}^g(1/R; m_1, \dots, m_N), \quad (\text{A.7})$$

where $\tilde{\Gamma}$ is an open string path integral with windings m_j along the boundaries B_j :

$$\begin{aligned} \tilde{\Gamma}^g(1/R; m_j) &= \sum_n \int_{n_B \cdot d\tilde{X}=0} \mathcal{D}' \tilde{X}_{nm} e^{-\frac{1}{4\pi} \int (d\tilde{X}_{nm})^2} \\ \oint_a d\tilde{X}_{nm} &= \frac{2\pi}{R} n_a \quad ; \quad \oint_{B_j} d\tilde{X}_{nm} = \frac{2\pi}{R} m_j. \end{aligned} \quad (\text{A.8})$$

In these expressions we have introduced an extra winding number $m_1 = -\sum_{j=2}^N m_j$ to obtain more symmetric looking formulas. If all $y_j = y$ coincide, then $\Gamma^g(R, y)$ is independent of y and we have the standard partition function with the compact volume factored out: $\Gamma(R, y) = \mathcal{Z}(R, y)/2\pi R$. On the right hand side of the duality equation (A.7) we have an open string vacuum amplitude with the zero mode also factored out. If we wish

to restore the zero mode we can write $\tilde{\mathcal{Z}}(1/R) = \frac{2\pi}{R}\tilde{\Gamma}(1/R)$. Now, in terms of $\tilde{\mathcal{Z}}$ the R power counting reads

$$\Gamma^g(R, y) \sim R^{-2g-N+2} \tilde{\mathcal{Z}}^g(1/R).$$

Since this surface comes with a coupling constant power $\sim \lambda^{2g+N-2}$, we have the usual mapping to the dual coupling $\tilde{\lambda} = \lambda/R$. An obvious generalization of the preceeding remarks includes connected world sheets with several combinations of Dirichlet and Neumann boundaries along the same coordinate, corresponding to perpendicular D-branes in the target space: a pair of order (p, p') is T-dual to a pair $(p-1, p'+1)$.

A.2. Non compact case

An interesting situation is the case of a non-compact isometry, which formally corresponds to the limit $R \rightarrow \infty$. The previous local derivation of the duality goes through except for the fact that now there are no sums over windings n_a or periodic translations m_j . As a result, the factors (A.5) and (A.6) do not appear, and the periods of the dual field remain unquantized. This also follows from (A.7) and (A.8) defining $\ell = n/R$ and $p = m/R$ and taking the large radius limit, with the result

$$\Gamma^g(\infty; y_j) = \text{const.} \times \int \prod_{j=1}^N dp_j e^{-ip_j y_j} \delta\left(\sum p_j\right) \int d\ell \tilde{\Gamma}^g(\ell_a; p_j), \quad (\text{A.9})$$

where $\tilde{\Gamma}^g(\ell_a; p_j)$ is the standard open string path integral with the boundary conditions

$$\oint_a d\tilde{X}_{\ell p} = 2\pi\ell_a \in \mathbf{R}; \quad \oint_{B_j} d\tilde{X}_{\ell p} = 2\pi p_j \in \mathbf{R}.$$

These unusual boundary conditions correspond to a continuous spectrum of winding modes, and thus represent the zero radius theory. A review of general facts about non-compact T-duality can be found in Appendix B.

A.3. Chan Paton factors

Returning to (A.8), we notice that the phase factors modulating the winding sums resemble insertions of Chan-Paton factors:

$$e^{-iy_j m_j/R} = e^{-\frac{iy_j}{2\pi} \oint_{B_j} d\tilde{X}_{nm}}.$$

In order to make the relation precise, we must symmetrize over the world sheet boundaries, so that each boundary can be mapped to a particular D-brane, and we must include a combinatorial factor $1/N!$ for N boundaries attached to the same brane in the target space. The result is equivalent to an insertion of

$$\prod_{\text{boundaries}} \text{Tr} P \exp \left(i \oint A_\mu dx^\mu \right), \quad (\text{A.10})$$

where the connection is given by $A_i = 0$ and $A_0 = \text{diag}(-y_j/2\pi)$. This corresponds to a particular $U(N_B)$ open string background, where N_B is the maximum number of D-branes. In the context of unoriented strings we have to consider $SO(2N)$ or $Sp(2N)$ groups. In this case the eigenvalues of the connection come in pairs of opposite sign, which is compatible with the presence of an orientifold in the D-manifold⁶.

The combinatorial factor discussed above is very important, because it serves to distinguish between several world sheet boundaries attached to one D-brane, and the situation in which a number of D-branes lie at the same point. In the first case we have a factor $1/N!$, whereas we would find $1/N_1!N_2!\dots$, with $N = N_1 + N_2 + \dots$ in the second case. Another way to notice superposition of D-branes is to switch on gauge fields compatible with some symmetry breaking pattern set by the connection A_0 . Suppose that A_0 has a block form

$$A_0 = -\frac{1}{2\pi} \text{diag} (y_k \mathbf{1}_{n_k}),$$

where $\mathbf{1}_{n_k}$ is the n_k -dimensional unit matrix. This configuration breaks $U(N)$ to $U(n_1) \times \dots U(n_k) \times \dots$. The most general non-abelian configuration that respects this symmetry breaking pattern consists of gauge fields with an appropriate block structure. The corresponding Chan Paton factors take then the form

$$\prod_{\text{boundaries}} \sum_k e^{-\frac{i}{2\pi} y_k \oint dx^0} \text{Tr}_{U(n_k)} P \exp \left(i \oint A_j^{(k)} dx^j \right).$$

After T-duality we find a path integral with Dirichlet conditions at y_k and non abelian Chan-Paton factors W_k corresponding to the $d-1$ Neumann coordinates. By construction

⁶ This pairing of eigenvalues is the origin of the 16 D-branes in the dual of type I. As we pointed out before, the absence of a $U(1)$ factor in this case translates into the rigid character of the orientifold.

these factors correspond to a world-volume $U(n_k)$ gauge theory. So we find the T-duality implementation of Witten's observation that n D-branes with an overall world-volume gauge group $U(1)^n$ produce a symmetry enhancement to $U(n)$ in the coincidence limit.

An interesting open problem concerns the T-duality transformation of the most general non-abelian Chan-Paton factor, where we may have a non trivial interaction between the A_0 and A_j components of the connection, due to the path ordering prescription. In view of the preceeding remarks, it is clear that this is related in the T-dual picture to the interactions between proximum D-branes (see [21]).

Appendix B. Remarks on non compact T-duality

In this appendix we recall some elementary facts about T-duality with respect to non-compact isometries.

It should be stressed from the start that in this case the dual theory is not “smooth”, at least interpreted as a sigma-model. We shall have more to say about this towards the end of the section. At any rate, if properly interpreted, the dual theory is *exactly* equivalent to the original one and, indeed, the involutive property can be explicitly checked.

As stated in appendix A, the main difference with the compact case lies in the unquantized holonomies of the dual field in a higher genus surface. This is a very general consequence of Kramers-Wannier duality in the cutoff theory. Consider for example a lattice gaussian model in a discretized Riemann surface:

$$\Gamma_{\text{discrete}}^g = \prod_{i=1}^{N_0-1} \int_{\mathbf{R}} \frac{dX_i}{2\pi} \prod_{(ij)=1}^{N_1} e^{-\frac{1}{4\pi}(X_i-X_j)^2},$$

where N_0 is the number of sites in the lattice and N_1 the number of links. They are related to the number of faces by the Euler theorem $N_2 = 2 - 2g - N_0 + N_1$. Keeping track of all factors of π , etc., the same manipulations that were introduced in appendix A lead to [20]:

$$\tilde{\Gamma}_{\text{discrete}}^g = \prod_{a=1}^{2g} \int_{\mathbf{R}} d\ell^a \prod_{I=1}^{N_2-1} \int_{\mathbf{R}} \frac{dX_I}{2\pi} \prod_{(IJ)=1}^{N_1} e^{-\frac{1}{4\pi}(X_I-X_J+2\pi\ell^a\epsilon_{IJ}^a)^2}.$$

Here I denotes sites in the dual lattice with $N_0^* = N_2$ vertices. The discrete one-forms $\epsilon_{IJ}^a = \pm 1$ when the link (IJ) has a positive (negative) intersection with a homologically non trivial path in the direct lattice, zero otherwise. There is some arbitrariness in the continuum limit, because we can absorb ultralocal functionals into the measure. Also, the one forms ϵ_{IJ}^a naively become delta-function distributions localized over the path a . Using the freedom to redefine $X_I - X_J$ by an exact one form, it is more convenient to adopt the prescription

$$X_I - X_J + 2\pi\ell^a \epsilon_{IJ}^a \rightarrow dX + 2\pi\ell^a h^a,$$

with h^a a basis of harmonic forms with normalized intersections $\oint_a h^b = \int h^a \wedge h^b = \delta^{ab}$ (these functions are linear in the upper half plane). The result is

$$\tilde{\Gamma}^g = \int_{\mathbf{R}^{2g}} d\ell \int \mathcal{D}' X_\ell e^{-\frac{1}{4\pi} \int (dX_\ell)^2}, \quad (\text{B.1})$$

where $\oint_a dX_\ell = \oint_a (dX + 2\pi\ell_a h^a) = 2\pi\ell_a$. In fact, we can give a direct proof in the continuum if we notice that the action in (B.1) factorizes as

$$\int (dX + 2\pi\ell \cdot h)^2 = \int (dX)^2 + 4\pi^2 \ell^a G_{ab} \ell^b,$$

where $G_{ab} = \int h_a \wedge * h_b$. One can then explicitly perform the integral over ℓ with the result

$$\int d\ell \Gamma_\ell = \Gamma^g \int d\ell e^{-\pi\ell^a G_{ab} \ell^b} = \det^{-1/2}(G) \Gamma^g.$$

With the standard normalization for h^a the matrix G can be inverted by a symplectic transformation (winding-momenta exchange), so the determinant is one and we have established $\tilde{\Gamma}^g = \Gamma^g$. This means that, at the level of the partition function, the continuous holonomies are quite harmless, because they are easily factorized. The situation is more complicated for correlation functions, where local vertex operator insertions of momentum q are mapped into vortices of charge q . The dual correlator is a frustrated partition function with local boundary conditions $\oint_{z_0} d\tilde{X} = 2\pi q$, where z_0 is the world sheet location of the operator. If we work in dual variables, with vortex correlation functions, the continuous holonomies around handles are necessary to understand world sheet factorization.

One could also factorize the classical part of the correlator, saturating the path integral with classical trajectories $X_{c\ell}$ or $\tilde{X}_{c\ell}$ which locally around punctures look like

$$X_{c\ell}(z, \bar{z}) \sim i q (\log(z - z_0) + \log(\bar{z} - \bar{z}_0))$$

$$\tilde{X}_{c\ell}(z, \bar{z}) \sim i q (\log(z - z_0) - \log(\bar{z} - \bar{z}_0)).$$

We can also arrive at the same result by explicitly taking the $R \rightarrow \infty$ limit of the compact case. The partition function at genus g is given by

$$\mathcal{Z}^g(R) = \sum_{n \in \mathbf{Z}^{2g}} \mathcal{Z}_n^g(R)$$

$$\mathcal{Z}_n^g(R) = \int \mathcal{D}X_n e^{-\frac{1}{4\pi} \int (dX_n)^2},$$

where $\oint_a dX_n = 2\pi R n_a$. Fixing the translational zero mode: $\int \mathcal{D}X = 2\pi R \int \mathcal{D}'X$ we can define $\Gamma^g(R) = \mathcal{Z}^g(R)/2\pi R$. Now, T-duality is the statement

$$\Gamma^g(R) = \text{const.} \times R^{-2g} \Gamma^g(1/R).$$

In the large radius limit no windings survive, so

$$\lim_{R \rightarrow 0} \Gamma^g(1/R) = \Gamma_0^g(\infty) = \lim_{R \rightarrow 0} \sum_n R^{2g} \Gamma_n^g(R).$$

Finally, defining $\ell = nR \in \mathbf{R}^{2g}$, and taking the limit, we arrive at (B.1):

$$\Gamma_0^g(\infty) = \int_{\mathbf{R}^{2g}} d\ell \Gamma_\ell^g.$$

So, our non-compact dual is the same as the zero radius limit, with the dilaton shift reabsorbed (note that the rescaling factor R^{-2g} was needed to transform the discrete winding sum into a continuous integral). The occurrence of the continuous holonomies $\ell \in \mathbf{R}^{2g}$ can be interpreted in two ways. We can say that we do not have a sigma model any more, because the embeddings are discontinuous in terms of the \tilde{X} field living in \mathbf{R} . We can also interpret the holonomies as a continuum of twisted sectors. After all, the windings build the twisted sector of the orbifold $\mathbf{R}/2\pi R\mathbf{Z}$ and, as R goes to zero, the winding spectrum becomes continuous. Furthermore, the invariant part of the spectrum (the momentum modes) disappears. So we can regard the non-compact dual as a sigma model in the “orbifold” [22]

$$\lim_{R \rightarrow 0} \frac{\mathbf{R}}{2\pi R\mathbf{Z}} \equiv \frac{\mathbf{R}}{0}.$$

Appendix C. Canonical Transformations in the operator formalism

When dealing with closed strings, it is well-known that T-duality can be interpreted as a canonical transformation [23]. Although for open strings the dual theory is somewhat exotic, it is still true that it can be formally interpreted as a canonical transformation in a big Hilbert space.

In the hamiltonian formalism T-duality corresponds to the following canonical transformation in the Hilbert space on the circle:

$$\tilde{\Psi}(\tilde{X}) = \int \mathcal{D}X e^{-iF(X, \tilde{X})} \Psi(X),$$

where Ψ and $\tilde{\Psi}$ are wave functionals and the integral is over a basis of the configuration space on the circle (not a path integral). The generating functional is very simple:

$$F(X, \tilde{X}) = \frac{1}{2\pi} \oint_{S^1} \tilde{X} dX.$$

In this appendix we explicitly construct the generating functional that implements T-duality for the boundary states of Neumann or Dirichlet character. The most general such boundary state (in the matter sector) was constructed in ref. [14]

$$|B\rangle_\eta = \int \mathcal{D}\bar{x} \mathcal{D}x \mathcal{D}\bar{\theta} \mathcal{D}\theta e^{-S_B} |\bar{x}, x\rangle \otimes |\bar{\theta}, \theta\rangle_\eta,$$

where $\eta = \pm$ labels the spin structure in the open string channel and, after T-duality, distinguishes branes from anti-branes. The configuration space coordinates satisfy ($m > 0$)

$$\begin{aligned} (a_m^{\mu\dagger} + \bar{a}_m^\mu - \bar{x}_m^\mu) |x, \bar{x}\rangle &= 0 \\ (a_m^\mu + \bar{a}_m^{\mu\dagger} - x_m^\mu) |x, \bar{x}\rangle &= 0 \\ (\theta_m^\mu - \psi_m^\mu + i\eta \bar{\psi}_m^{\mu\dagger}) |\theta, \bar{\theta}\rangle_\eta &= 0 \\ (\bar{\theta}_m^\mu - \psi_m^{\mu\dagger} - i\eta \bar{\psi}_m^\mu) |\theta, \bar{\theta}\rangle_\eta &= 0, \end{aligned}$$

where the index m in ψ_m runs over the integers in the R-R sector of the closed string, and half integers in the NS-NS sector. The relation of a_m^μ with the standard notation of bosonic oscillators is $\alpha_m^\mu = -i\sqrt{m} a_m^\mu$. The normalized “position” eigenstates are

$$|x, \bar{x}\rangle = \exp\left(-\frac{1}{2} \bar{x} \cdot x - a^\dagger \cdot \bar{a}^\dagger + a^\dagger \cdot x + \bar{x} \cdot \bar{a}^\dagger\right) |0\rangle$$

$$|\theta, \bar{\theta}\rangle_\eta = \exp\left(-\frac{1}{2}\bar{\theta} \cdot \theta + i\eta \psi^\dagger \cdot \bar{\psi}^\dagger + \psi^\dagger \cdot \theta - i\eta \bar{\theta} \cdot \bar{\psi}^\dagger\right) |0\rangle.$$

The dot product is defined by

$$\bar{x} \cdot x = \sum_{\mu=1}^D \sum_{m>0} \bar{x}_m^\mu x_m^\mu,$$

and the position states are conveniently normalized to one, so that the free (Neumann) boundary states corresponding to $S_B = 0$ are

$$|\text{Neumann}\rangle_\eta = \exp\left(a^\dagger \cdot \bar{a}^\dagger - i\eta \psi^\dagger \cdot \bar{\psi}^\dagger\right) |0\rangle.$$

The zero modes in the R-R sector carry a representation of the $SO(1, 9)$ Clifford algebra, and the vacuum satisfies

$$(\psi_0^\mu - i\eta \bar{\psi}_0^\mu) |0\rangle_\eta = 0.$$

Under T-duality we have $\bar{a}^\dagger \rightarrow -\bar{a}^\dagger$ and $\bar{\psi}^\dagger \rightarrow -\bar{\psi}^\dagger$ (for simplicity of notation, we consider here the duality transformation of all coordinates, so that we are dealing with the D-instanton). We know that this is generated by a canonical transformation with generating function of the form $F(q, \tilde{q})$, where \tilde{q} are the dual variables. It is very easy to find such function by simple gaussian integral manipulations requiring

$$|\tilde{q}\rangle_{\text{dual}} = \int \mathcal{D}q \, e^{-iF(\tilde{q}, q)} |q\rangle.$$

The sign of the anti-holomorphic components is inverted with the following generating function:

$$F(x, \bar{x}, \theta, \bar{\theta}; \tilde{x}, \tilde{\bar{x}}, \tilde{\theta}, \tilde{\bar{\theta}}) = \frac{i}{2} \left(\bar{x} \cdot \tilde{x} - \tilde{\bar{x}} \cdot x + \bar{\theta} \cdot \tilde{\theta} - \tilde{\bar{\theta}} \cdot \theta \right) + \text{constant}.$$

Introducing mode expansions

$$X(\sigma) = q + \frac{1}{\sqrt{2}} \sum_{m>0} \frac{1}{\sqrt{m}} (x_m e^{-im\sigma} + \bar{x}_m e^{im\sigma})$$

$$(\psi + i\eta \bar{\psi})(\sigma) = \psi_\eta(\sigma) = \frac{1}{\sqrt{2}} \sum_{m>0} (\theta_m e^{-im\sigma} + \bar{\theta}_m e^{im\sigma}),$$

and similarly for the dual fields, we obtain the final result

$$F(X, \tilde{X}; \psi_\eta, \tilde{\psi}_\eta) = \frac{1}{2\pi} \oint_{S^1} \left(\tilde{X} dX - i\tilde{\psi}_\eta \psi_\eta \right),$$

where the fermionic terms are restricted to the non-zero modes. In general, the zero mode part of the dual state is determined by the constraint:

$$(\tilde{\psi}_0 + i\eta \tilde{\tilde{\psi}}_0)^{\text{tr}} |0\rangle_\eta = 0,$$

where “tr” stands for the transverse coordinates to the p -brane (all the coordinates for the case of the D-instanton). An analogous generating function can be written in the super-ghost sector.

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